PHYS1141: Nuclei & Radioactivity

Problem Set 1 -- Solutions

1. How many grams of matter would have to be converted totally to energy in order to run a 100-W light bulb for 1 year?

Solution: Energy required = (100 W)(24×3600×365 s) = 3.15E+9 J. So mass required

\[ \text{energy}/c^2 = (3.15E+9 J)/(3.00E+8 m/s)^2 = 3.5E–8 kg = 35 \text{ µg}. \]

2. Suppose a spacecraft of rest mass 20 tonnes (1 tonne = 1000 kg) is accelerated to 0.25c. (a) How much kinetic energy would it have? (b) If you used the classical formula \( K = mv^2/2 \), by what percentage would you be in error?

Solution: (a) Total energy \( E = mc^2 = \gamma m_0 c^2 \); kinetic energy (total energy less rest mass energy)

\[ K = E - m_0 c^2 = (m - m_0)c^2 = (\gamma - 1)m_0 c^2. \]

In this case, \( \gamma = 1/\sqrt{[1 - (0.25)^2]} = 4/\sqrt{15} = 1.0328 \). Hence

\[ K = (\gamma - 1)m_0 c^2 = (3.28E–2)(20E+3 kg)(3.00E+8 m/s)^2 = 5.904E+19 J = 5.9E+19 J. \]

(b) The non-relativistic formula: \( K = m_0 v^2/2 = (1/2)(20E+3 kg)(0.25 \times 3.00E+8 m/s)^2 = 5.625E+19 J \).
This is too low by 0.279E+19 J; ie by 100(0.279/5.904)% = 4.7%; i.e. about 5% low.

3. In the Fermilab synchrotron, protons of 400 GeV total energy are kept revolving in a circle of radius 1.0 km. What magnetic field strength is needed for this? The proton rest mass energy is 0.938 GeV.

Solution: A proton (charge e & relativistic mass \( m \)) is following a circular orbit (speed \( \nu \) and radius \( r \)) around magnetic field lines. The circle, and hence the velocity, must be perpendicular to the field, so the magnetic force has magnitude \( evB \). The magnetic force is the centripetal force, which has magnitude \( mv^2/r \). Therefore

\[ mv^2/r = evB, \text{ which gives } r = mv/eB, \]

the same formula as in the non-relativistic case, except that the relativistic mass appears here.

The protons have \( E = 400 \text{ GeV} \); since this is much bigger than their rest mass energies (about 0.9 GeV), they are highly relativistic, so \( \nu = c \) to high accuracy; hence \( r = mc/eB \). Thus:

\[ B = mc/e\nu = mc^2/\nu c = E/\nu c \text{ since } E = mc^2. \]

Hence: \( B = [(400E+9 \text{ eV})(1.60E–19 \text{ J/eV})]/[(3.00E+8 \text{ m/s})(1.60E–19 \text{ C})(1.0E+3 \text{ m})] = 1.3 \text{ T} \).
{ Note: with \( m_p \equiv \text{proton rest mass, the Lorentz factor} \}

\[ \gamma = (400 \text{ GeV})/(m_pc^2) = (400 \text{ GeV})/(0.938 \text{ GeV}) = 426. \}
4. (a) Determine the density of nuclear matter (in kg m\(^{-3}\)) and show that it is essentially the same for all nuclei. (b) The Earth has mass \(6.0 \times 10^{24}\) kg and radius 6400 km. What would its radius become if the Earth were condensed to the density of nuclear matter? (c) The Sun has mass \(2.0 \times 10^{30}\) kg and radius \(7.0 \times 10^{5}\) km. What would its radius become if the Sun were condensed to the density of nuclear matter? (d) What would be the radius of a \(^{238}\text{U}_{92}\) nucleus if it had the density of the Earth?

**Solution:** (a) Let \(m_N\) denote the nucleon mass of 1.67E–27 kg. If we neglect the small (0.1%) neutron-proton mass difference, a nucleus consisting of \(A\) nucleons has mass \(A m_N\). Its radius \(r = 1.2A^{1/3}\) fm. Since volume = \((4\pi/3)r^3\), the density of the nucleus is (note that \(A\) cancels):

\[
(1.67E–27 \text{ kg})/[(4\pi/3)(1.2E–15) \text{ m}^3] = 2.3E+17 \text{ kg m}^{-3}
\]

(which is 2E+5 tonne /mm\(^3\)). This density is independent of \(A\), and so can be described as the density of nuclear matter.

(b) The mass of the Earth is 5.98E+24 kg. So, if its density were that of nuclear matter, then its volume would be \((5.98E+24 \text{ kg})/(2.3E+17 \text{ kg m}^{-3}) = 26E+6 \text{ m}^3\). From volume \(V = (4\pi/3)r^3\), we have \(r = [(3/4\pi)V]^{1/3}\). So the Earth’s radius would become \([(3/4\pi)26E+6 \text{ m}^3]^{1/3} = 180 \text{ m}\).

(c) The mass of the Sun is 2.0E+30 kg. So, if its density were that of nuclear matter, then its volume would be \((2.0E+30 \text{ kg})/(2.3E+17 \text{ kg m}^{-3}) = 8.7E+12 \text{ m}^3\). From volume \(V = (4\pi/3)r^3\), we have \(r = [(3/4\pi)V]^{1/3}\). So the Sun’s radius would become \([(3/4\pi)8.7E+12 \text{ m}^3]^{1/3} = 13 \text{ km}\).

(d) The mean density of the Earth is 5.5E+3 kg m\(^{-3}\). The mass of a \(^{238}\text{U}_{92}\) nucleus is (approx) 238\(m_N\). If its density were that of the Earth, then its volume would be \((238 \times 1.67E–27 \text{ kg})/(5.5E+3 \text{ kg m}^{-3}) = 72E–30 \text{ m}^3\). Hence the radius of this nucleus would become \([(3/4\pi) \times 72 \times 10^{-30}]^{1/3} \text{ m} = 0.26 \text{ nm}\) (which is similar to the sizes of atoms).

5. sees over page

6. Evaluate the binding energy (in eV units) of a helium nucleus -- ie the energy required to break it into its constituents, two protons and two neutrons. The rest masses of a proton, a neutron and a helium nucleus are, respectively, 1.00783 u, 1.00867 u and 4.00260 u. One u denotes one unified atomic mass unit, defined to be precisely one-twelfth of the mass of a neutral \(^{12}\text{C}_{6}\) atom; 1 u is equal to 1.6605 \times 10^{-27} kg, which is equivalent to 931.50 MeV of energy.

**Correction:** the rest masses quoted in this problem are actually for a neutral H atom, a neutron, and a neutral He-4 atom.

**Solution.** The sum of the masses of the separated constituents in the form of two neutral H atoms plus two neutrons is

\[
2m(^1\text{H}_1) + 2m_n = 2 (1.00783 + 1.00867) \text{ u} = 4.03300 \text{ u}.
\]

This exceeds the mass of a neutral He-4 atom (4.00260 u) by 0.03040 u. [The two electrons have been included after and before separation, so their masses have cancelled out.]

Since a mass of 1 u is equivalent to an energy of 931.50 MeV, the binding energy of the He nucleus is 0.03040 \times 931.50 \text{ MeV} = 28.3 \text{ MeV}.
5. Calculate the kinetic energy, in keV, for an electron travelling at speeds given as follows, using both the classical and relativistic KE formulas:

(a) \( v/c = 0.1 \);  (b) \( v/c = 0.3 \);  (c) \( v/c = 0.5 \);  (d) \( v/c = 0.9 \);  (e) \( v/c = 0.99 \).

In each case, state whether you consider the electron to be non-relativistic, moderately relativistic, or highly relativistic.

**Solution:** With \( m_e \) denoting the electron rest mass,

\[
K' = \left(\frac{1}{2}\right)m_e v^2 \quad \text{and} \quad K \equiv (\gamma - 1)m_e c^2 \quad \text{with} \quad \gamma(v) \equiv \frac{1}{\sqrt{1 - v^2/c^2}}.
\]

are the non-relativistic & relativistic kinetic energy formulas. That is

\[
K'/m_e c^2 = \left(\frac{1}{2}\right)(v^2/c^2) \quad \text{and} \quad K/m_e c^2 = \gamma - 1, \quad \text{with} \quad m_e c^2 = 511 \text{ keV}.
\]

(a) For \( v/c = 0.1 \): \( \gamma = 1.005,038 \);

\[
K' = \left(\frac{1}{2}\right)(0.01)(511 \text{ keV}) = 2.555 \text{ keV}; \quad K = (0.005,038)(511 \text{ keV}) = 2.574 \text{ keV}.
\]

So \( K' \) is in error by 0.7%. The particle is only slightly relativistic; we can say it’s non-relativistic to a good approximation.

(b) For \( v/c = 0.3 \): \( \gamma = 1.048,285 \);

\[
K' = \left(\frac{1}{2}\right)(0.09)(511 \text{ keV}) = 22.995 \text{ keV}; \quad K = (0.048,285)(511 \text{ keV}) = 24.67 \text{ keV}.
\]

So \( K' \) is in error by 7%. The particle is moderately relativistic. The KE is becoming a significant fraction of the rest-mass energy.

(c) For \( v/c = 0.5 \): \( \gamma = 1.154,701 \);

\[
K' = \left(\frac{1}{2}\right)(0.25)(511 \text{ keV}) = 63.88 \text{ keV}; \quad K = (0.155)(511 \text{ keV}) = 79.05 \text{ keV}.
\]

So \( K' \) is in error by 19%. The particle is moderately relativistic. Its KE is a significant fraction of its rest-mass energy.

(d) For \( v/c = 0.9 \): \( \gamma = 2.29416 \);

\[
K' = \left(\frac{1}{2}\right)(0.81)(511 \text{ keV}) = 207 \text{ keV}; \quad K = (0.29416)(511 \text{ keV}) = 661 \text{ keV}.
\]

The particle is relativistic. Its KE is of the same order of magnitude as its rest-mass energy. The non-relativistic KE formula is very wrong.

(e) For \( v/c = 0.99 \): \( \gamma = 7.089 \);

\[
K' = \left(\frac{1}{2}\right)(0.980)(511 \text{ keV}) = 250 \text{ keV}; \quad K = (6.089)(511 \text{ keV}) = 3,110 \text{ keV}.
\]

The particle is quite highly relativistic. Its kinetic energy is 6.1 times as big as its rest-mass energy.